Cube duplication, also known as the "Delian problem" was, along with the angle trisection and circle squaring, one of the three classical problems of Greek geometry whose solutions were sought using only compass and straightedge.

According to tradition, in 427 BC a violent plague decimated the inhabitants of Athens causing, even, the death of its prestigious leader Pericles. Dismayed by this enormous loss, the inhabitants sent a delegation to the Oracle of Apollo in the Temple of Delphi to know how to end the disease. The Oracle advised doubling the size of the cubic altar to Apollo. Ingeniously, the Athenian doubled the length of the edges: as a result the new altar was actually eight times as big as the old one and the plague continued to spread unabated.

As for the Temple of Delphi it is worth clarifying that, although it was considered by the contemporaries as the spiritual centre of paganism of the time (the "navel of the world"), it had nothing to do with the deleterious and corrupt image imposed by Christianity during almost 2,000 years. On the contrary, we know that, carved into the Temple, were the phrase: gnōthi seautón, which means "know thyself". Furthermore, in the Phaedrus, Plato points to the Temple of Delphi as the source of many of the greatest goods that come to human beings, and states that the Temple is a formidable “communication tool” between mortal creatures and Divinity.

There is no doubt that in Delphi the Greeks concretely expressed their love for knowledge, albeit different from the modern concept of Science: a knowledge directed mainly to the metaphysical wisdom. In the Phaedrus, Socrates says that it would be ridiculous to investigate things that are strange to him, not having he yet attained the capacity to know himself. On the other hand, Plato underlines the extreme importance of self-knowledge warning, at the same time, that the soul (psyché) is the essence of man and that it is imperative, rather than the body itself, to take good care of the soul, since a life without self-knowledge would not be worth living. This philosopher also affirms that the inner search is "a service to God" because, in so doing, we obey his will [1]. We can not rule out a priori that the response given by the Oracle had something to do with the epithet gnōthi seautón and was therefore related to some ritual already existing in the Temple [2].
The first mathematician to attempt a solution of the Delian problem was Hippocrates of Chios (460-380 BC), disciple of Pythagoras. Hippocrates reduced a problem of solid geometry to one of flat geometry which, however, could not be solved by using only ruler and compass. Nonetheless, this intermediate step paved the way for the development of new geometric techniques, in particular that involving two means between a line segment and another with twice the length. The question of the proportional means was held by the Pythagoreans as very important, since it was directly related to the cosmological view of their community [3].

Other interesting solutions came from Archytas (430-360 BC), Menaechmus (380-320 BC), Nicomedes (250-180 BC), Diocles (240-180 BC), and Eratosthenes (275-195 BC); but there are other outstanding solutions, equally valid, proposed by Plato, Apollonius, and Philo of Byzantium. In the nineteenth century, mathematicians Ruffini, Abel and Galois demonstrated the impossibility of solving the problem using only ruler and compass. Finally, in 1837, Pierre Wantzel proved that the cube root of 2 is not constructible because it is not a Euclidean number, that is to say, a number which can be obtained by repeatedly solving the quadratic equation.

However, at the end of that same century, three Italian mathematicians: Gaetano Buonafalce, Giuseppe Vargiu and Gaetano Boccali presented original and creative two-dimensional solutions that, although not rigorously accurate, at least gave good approximate results (error <0.1%) [4]. In particular, the solution proposed by Gaetano Boccali in 1884 started from the regular starred decagon, a figure of great importance to the Pythagoreans, since it is from the decagon that one can construct the pentagram (or pentalfa), a favourite symbol of the adherents of Pythagoras and the esoteric scholars throughout the centuries. The figure below shows a starred decagon composed of two pentagrams.

From an algebraic point of view the doubling of the cube is so simple that a twelve-years-old student could find the solution in less than a minute. For, being the volume of a cube given by the cube of the length of its edge, it is evident that if the volume is 2, then the length of the side will be $\sqrt[3]{2}$. We must take into account, however, that twenty-four centuries ago Greek mathematicians were not only ignorant of analytic geometry but also of algebra, and zero was unknown to them. Therefore, mathematical problems were solved by geometry: a relatively complex task with flat figures, but an extremely complicated one in the case of solid figures.
Among the solutions listed above, that of Archytas is the most notable. It is a skilful construction in three dimensions, where the solution is represented by a point identified by the intersection of three surfaces of revolution, i.e.: a cylinder, a cone, and a torus. The next image, generated by a computer, shows the position of the three solids:

![Image of three solids: green torus, blue cylinder, red cone]

The toro is painted green, the cylinder blue and the cone red. The solution, beside being absolutely rigorous and extremely beautiful, shows that the Pythagorean School already used the concept of locus and made use of spatial geometry [5]. Archytas, of course, could not rely on appropriate software, colored markers, transparent sheets, graph paper or other modern tools for drawing. Worse, the analytical geometry and the Cartesian coordinates were totally unknown to him. We deduce, therefore, that this excellent ancient mathematician dominated to a very high degree the capacity for abstraction, a dowry typical of the great scientists, the best philosophers, and the most excellent poets.

We note, on the other hand, that the solutions presented by Greek geometers of the time, although brilliant, were apparently restricted to the domain of profane Geometry. Even so, by going into the details of these solutions, we discover how they obeyed a deeper and more mystical conception of the ancient pagan cosmogony. This view had its foundation in Platonic metaphysics according to which, as we read in the Timaeus, "two things can not be joined together properly, without a third." Thus, starting from the Unit (symbolized by a circle) containing everything, one gets the Dyad where the Unit separates and differentiates itself through a process known as polarization; the two parts remain united by the action of the Demiurge.
In order to graph this fundamental concept, it is necessary to draw two circles of radius 1 whose respective centers are spaced apart from one unit, so that they can interact mutually. The intersection of the two circles symbolizing, respectively, matter and spirit generates a common area in the form of a fish bladder (Vesica Piscis) or "almond", which represents the Demiurge, the factor of the material Universe.

Geometric constructions on the “almond” generate some important irrational numbers such as $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$. Also, always from this fundamental figure, we can construct important figures of the flat geometry, namely: triangle, square, pentagon, hexagon, and decagon. For now, we will focus on the squares inside the Dyad.

Let’s consider two centres $C$ and $C'$ from which we draw two segments perpendicular to the axis $A-A$ and tangent to the circumferences in $C$ and $C'$. These new segments intercept the circumferences in $D$, $E$, and $F$, $G$, respectively. In such a way, we obtain two squares whose sides are 1, whereas,
their diagonals, for the theorem of Phytagoras, are $\sqrt{2}$. Of course, the area of both squares is 1. Therefore, it is evident that the surface of the parallelogram $DEGF$ is 2.

Let's now imagine a rotation of the Dyad of 180° around its axis $Z-Z$. The following solids are generated:

- A right circular cylinder with diameter 1 and height 2;
- A spindle, whose maximum width is 1 and height is $\sqrt{3}$;

A spindle torus, which is a torus whit a spindle inside it. The figure below shows the lower half of a spindle torus cut by a plane passing for the axis $A-A$, and perpendicular to the axis $Z-Z$. 
Its implicit quartic equation in Cartesian coordinates is:

\[
\left(\frac{1}{2} - \sqrt{x^2 + y^2}\right)^2 + z^2 = 1
\]

where, putting \( x = 0 \) and \( y = 0 \) one has:

\[ z = \pm \frac{\sqrt{3}}{2} \]

namely, half the height of the spindle and of the \textit{Vesica Piscis}.

One of the most interesting properties of any torus is that, by cutting it obliquely through the centre at a special angle, one obtains two interlocking "Villarceau circles", that is, a new \textit{Vesica Piscis}.

Let’s now consider the spindle (\textit{Vesica Piscis} in three dimensions); we observe that this solid is exactly contained in a parallelepiped whose four largest rectangular faces are tangent (for \( z = 0 \)) to the surface of the spindle and parallel to its major axis Z-Z, while the two smaller, square faces are perpendicular to the axis. Being the parallelepiped composed by two cubes, each of them volume 1, its total volume is 2.
Now, it is worthwhile introduce the geometric mean, a tool discovered by the Pythagoreans and formalised by Archytas. If the sides of a rectangle measure $a$ and $c$ respectively, its area will be the $ac$ product. Consequently, the side of a square having the same area will be:

$$l = \sqrt{ac}$$

that is, the geometric mean of the two sides $a$ and $c$. This, of course, holds for a two-dimensional object. In three dimensions the volume of a parallelepiped whose sides measure $a$, $b$, and $c$ is the same as that of a cube whose side $l$ measures:

$$l = \sqrt[3]{abc}$$

Since in this case the volume is 2, the double-volume cube side will be:

$$l = \sqrt[3]{2}$$

The irrational algebraic number 1.2599 ... is also known as Delian constant. This solution, however, does not completely solve the problem, in the sense that it still does not provide a method for using only ruler and compass. Nonetheless, it shows how, starting from the Dyad, the three solids used by Archytas are generated, and it stands to reason to imagine that this genius knew well the Platonic Dyad because, besides being friend of the illustrious philosopher, he saved his life in one of the trips of Plato to Sicily.

Another ingenious solution to the problem of cube duplication comes from Menaechmus. He showed how the unknown side of the cube with twice the volume could be obtained either as a point of intersection between two parabolas or as a point of intersection between a parabola and a hyperbola. In geometry, the parabola, the circle, the ellipse, and the hyperbola are four different types of conic sections obtained as the intersection of the surface of a cone with a plane.

The reason why it is necessary to resort to the conic lies in the intuition, attributed to Hippocrates of Chios, according to which the Delian problem could be solved by inserting two means $x$ and $y$ between two known segments ($a$ and $b$), that is:

$$a : x = x : y = y : b$$
In our case, \( a = 1 \) and \( b = 2 \) are, respectively, the volumes of the two cubes. Rewriting the proportion in the form of an algebraic equation we obtain:

\[
\frac{1}{x} = \frac{x}{y} = \frac{y}{2}
\]

From the first equality we have:

\[
y = x^2
\]

which is the equation of a parabola with vertex at the origin and axis that coincides with the vertical axis of the ordinates. From the second equality we have:

\[
x = \frac{1}{2}y^2
\]

which represents another parabola perpendicular to the previous one. Equating the extremes we obtain the equation of a hyperbola:

\[
y = \frac{2}{x}
\]

By solving any of the systems formed by these three equations the result is always the same:

\[
x = \sqrt{2}
\]

The figure below shows the graph of the two parabolas (left) and a parabola with the hyperbola (right) that intersect at the same point \( A \) corresponding to the solution of the problem. 

It is obvious that, without the tools offered by algebra, this solution was, for Menaechmus, much more complicated than the one we have just presented.

Now, another legitimate question arises: do the conic sections also have their origin in \textit{Vesica Piscis}? The answer is yes!
Let's return to the Dyad and draw its two axis. Now, from A, we draw two segments that pass, respectively, through the centres $C$ and $C'$: what we get are the two similar equilateral triangles $ACC'$ and $AED$. The rotation of the larger triangle with respect to the $Z-Z$ axis generates the lower half of a cone while the continuation of the $EA$ and $DA$ segments above the Dyad generates, by rotation about the same axis, the upper half of a cone.

We have thus seen how the intersection of two circles, which represent the splitting of the Platonic Monad, generates the Dyad and the Vesica Piscis which, in turn, is the mother who generates all geometric figures. It is therefore proven that the foundation of Geometry in the Ancient World had a deeply spiritual nature.

Archytas, in its construction, did not use a spindle torus, but rather a horn torus, a cylinder of radius 1 eccentric with respect to the $z$-axis, and a horizontal cone (see the initial figure), whose equations are as follows:

\[
\begin{align*}
(x^2 + y^2 + z^2)^2 &= 4(x^2 + y^2) \\
x^2 + y^2 &= 2x \\
x^2 &= y^2 + z^2
\end{align*}
\]

From the latter equation we obtain:

\[y^2 = x^2 - z^2\]
By replacing $y^2$ in the first equation, one can eliminate the variable $z$ and, by means of the second equation, we substitute the binomial on the right side of the first equation with $2x$. Thus the first equation reduces itself to the following:

$$(x^2 + x^2)^2 = 8x$$

that is, $x^3 = 2$ whose solution is:

$$x = \sqrt[3]{2}$$

We close this work by drawing the reader's attention to the famous "eye" that occasionally appears in the centre of the triangle called the “Luminous Delta”, the Masonic name of the Pythagorean Tetraktys. It can be observed, for example, in the American dollar bill.

This "eye" has nothing to do with the Trinitarian "eye of the Father", for it was already known in Plato's time and can be obtained by drawing a circle of radius 1 inside the Vesica Piscis. The circumference measures exactly $\pi$ and the figure closely resembles a human eye where the pupil is nothing if not the centre of the circle.
While the rotation of the circle of radius 1, in relation to any one of its symmetry axis generates a sphere of diameter 1 contained, in turn, in a cube of volume 1, the rotation of the "eye" with respect to the axis $A-A$ generates a double spherical cap containing a central spherical bulge. It is a geometric shape very similar to that of the group of Spiral Galaxies, to which our Milky Way belongs.

Diogenes Laertius was right when, in the third century of our era, he wrote that "The Monad is the beginning of all things". So, Vesica Piscis is indeed related to the most important forms of our Universe, from the fundamental geometric figures to the galaxies, where stars, planets and intelligent beings are born.

References


